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Linear analysis of rapidly switched heat regenerators in counterflow

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Abstract

This paper is intended to provide an accurate analytical solution to the 1D differential equations modelling cyclic steady heat transfer processes in rapidly switched heat regenerators for any value of the flush ratio. The temperature solution for the fluid is initially given in an integral form along the path of a gas particle as a function of the matrix temperature for different space and time intervals. In particular, as a Lagrange system of reference is assumed, the above solution deals separately with gas particles of three possible types ('cold', 'hot' and 'internal') according to Organ's concept of independent flow regimes. Also, it accounts for the possible superposition of the so-called hot and cold zones of the regenerative matrix depending on the value of the flush ratio. Then, assuming a linear distribution for the matrix temperature, the fluid temperature may analytically be calculated. A closed-form expression for the regenerator effectiveness as a function of NTU and flush ratio is given. It provides a simple but accurate tool to estimate the regenerator effectiveness in rapid cyclic flow situations and the deriving results indicate that it is underestimated by the conventional regenerator theory.

Keywords: Heat regenerators; Rapidly switched; Flush phase; Cyclic operation; Counterflow; Regenerator losses; Lagrange system

1. Introduction

Heat regenerators are thermal energy storage devices where the processes of heat storage and heat retrieval are cyclically repeated and the hot and cold fluids usually flow in opposite directions (counterflow operation). Their performance during the cyclic steady operation depends on three dimensionless variables, for instance, N_{TU}, U and α. The number of transfer units N_{TU} and the utilization factor U were first introduced by Hausen [1,2, chapter 35]. The former is defined as the heat transfer coefficient – surface area of matrix product to the heat capacity flow rate of the gas (for instance, it was denoted by Hausen reduced length Λ). The latter specifies the ratio of the thermal capacity of the gas per pass to that of the matrix. (Actually Hausen used the reduced period Π, but since $U = \Pi/\Lambda$, this is only a minor change suggested by Johnson [3,4] with the advantage to not include the heat transfer coefficient.) As regards the flush ratio α, it was first introduced by Organ [5,6] and it is defined as the ratio of the time required for a gas particle to complete a regenerator traverse to the duration of a period, namely the '*blow*' or '*reverse*' periods. (As a matter of fact, it was formerly denoted N_{FL} and defined as $N_{\mathrm{FL}} = \alpha^{-1}$.) It accounts for the flush phase, i.e. the fact that some of the working fluid might not pass all the way through the regenerator but could remain (or be 'held up') inside the regenerator. Thus, the utilization factor may be written as $U = (\alpha\beta)^{-1}$, where β is the thermal capacity ratio specifying the heat capacity of a length unit of the matrix, $\rho_w c_w(A - A_f)$, to the heat capacity of a length unit of the gas, $\rho c_p A_f$.

In the regenerators used in blast and glass melting furnaces, in metallurgical and chemical processing industries (of fixed-bed type) [7,8] as well as in electrical power generating stations for air preheating, and in gas turbine power

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Nomenclature

A	cross-sectional area of the regenerator
A_f	cross-sectional area for flow
$c\rho$	gas heat capacity per unit of volume
$c_w\rho_w$	matrix heat capacity per unit of volume
h	heat transfer coefficient
L	regenerator length
n	nth operation cycle
N_{TU}	number of transfer units, $hLP/(\rho c_p\lvert u_j\rvert A_f)$
P	matrix wetted perimeter
t_{ij}	initial time for the jth period
t_j	time during the jth period
T, T_w	fluid and matrix temperatures
u_j	gas flow velocity during the jth period
U	utilization factor, $\rho c_p\lvert u_j\rvert A_f(\tau_0/2)/[\rho_w c_w(A-A_f)L]$
x	space coordinate (Euler system)
$x_j(t_j)$	law of motion of a gas particle during the jth period (Lagrange system)
x_{ij}	initial location of a gas particle for the jth period

Greek symbols

α	flush ratio, $2L/(\lvert u_j\rvert\tau_0)$
β	thermal capacity ratio, $c_w\rho_w(A-A_f)/(c_p\rho A_f)$
ε	effectiveness
ϑ	dimensionless temperature, $(T-T_k)/(T_h-T_k)$
ξ	location of the gas particle within the regenerator
τ_j	time corresponding to ξ during the jth period
τ_0	duration of an entire operation cycle

Subscripts

b	blow period
h	hot space
k	cold space
j	index (b or r)
r	reverse period
w	matrix

Superscripts

$+$	dimensionless (τ_0 for time and L for space)
h	hot particles
i	internal particles
k	cold particles

plants (of rotary type) [9,10], the time required for an element of gas to pass through the regenerator (L/u) is very short compared to the time of either period ($\tau_0/2$). These regenerators are in fact usually very large heat exchangers, some having spatial dimensions of up to 40 m and having unidirectional flow periods of many hours. This indicates that they are '*slowly switched*' ($\alpha \to 0$) and their performance may be well described by only two dimensionless variables, namely N_{TU} and U, as done by Hausen. Notice that the utilization factor U is finite as $\beta \to \infty$ in that the matrices employed in the engineering applications stated before are usually metallic.

In the regenerators used in Stirling prime movers [11,12], coolers [13], heat pumps [14] and cryo-coolers [15–17] (of fixed-bed type), the time required for a gas particle to pass through the regenerator is however approximately equal to the blow time (or reverse time). These regenerators are in fact considered large if their diameter exceeds 6 cm and unidirectional flow periods are more likely to be in the millisecond range. This indicates that they are '*rapidly switched*' (α = finite) and the effects of the flush phase have to be taken into account. Their performance may hence be well described by only two dimensionless variables, namely N_{TU} and α, in that the utilization factor U approaches zero (as $\beta \to \infty$), unless the matrix used in the heat regenerators is not metallic (β = finite). In such a case, in fact, only using all of the three dimensionless variables, namely N_{TU}, U (or β) and α, ensures an appropriate description of their operation, as done by Organ.

A full treatment of the 'rapidly switched regenerator problem' was further developed by Organ in a very authoritative book [18] running to 623 pages and looking back over almost 200 years. This treatment is based on a Lagrangian formulation and analyzes both transient and cyclic steady operations indicating that the 'regenerator problem' is essentially an extension of the well-known conjugate heat exchange problem [19]. The concept of *natural coordinates* [20] allows the *real particle trajectories* of the gas to be computed and, hence, the mixed Lagrange–Euler integration grid to be appropriately established. The physical picture of regenerator operation accounts for flow friction, space and time variable particle speed and heat transfer coefficient as well as cyclic temperature fluctuations of the matrix and variations in pressure with time and location [5,6,19,21]. The governing equations are numerically solved and charts of the effectiveness are given in Refs. [6,18, chapter 7].

On the other hand, Organ's early studies were concerning with the description of the dynamic flow characteristics of the complete regenerator in terms of a single variable – a complex admittance [22] – using the methods of linear wave theory [23]. Contrary to the regenerator thermal solution described above, where the gas temperature is computed under prescribed flow and pressure conditions, the wave methods permit pressure and velocity in function of time and location to be computed at prescribed temperature. Therefore, these methods are neither in competition with the thermal approach nor relevant to it. A first attempt to combine the pressure wave and thermal solutions is given by the same author in his last book concerning Stirling and pulse-tube cryo-coolers [17].

A further numerical thermal solution using a finite difference method was recently proposed by Ataer [24]. It accounts for the flush phase proposed by Organ but the analysis was limited to those few free-piston Stirling engines of ‘beta’ type (for example, thermo-mechanical generators [25]) where the circular annulus between the cylinder wall (fixed) and displacer wall (in reciprocating motion) serves as a regenerator. As regards the analytical solutions, they are few and far between since the rapidly switched regenerator problem is generally held to be complex and demanding. An exact analytical solution to the above problem modelling the complex nature of the oscillating and reversing flow circulations was however given by de Monte [26] in terms of two dimensionless variables, namely N_{TU} and α, using a Lagrange formulation for cyclic steady operation and neglecting matrix temperature oscillations. Recently, Finkelstein and Organ [27, chapters 11–12], have considerably simplified formulation of the regenerator thermal solution dispensing with all of the general cases initially analyzed by Organ [5,6,18–21], so providing an approximate expression for the regenerator effectiveness in the simple form of $\varepsilon = 1 - 1/N_{TU}$ which is valid for high values of N_{TU} and is independent of α.

As the analytical procedure proposed by de Monte [26] was limited to only small values of the flush ratio (i.e. up to 1), the objective of the present paper is to extend this procedure to cases characterized by $\alpha > 1$. It would indicate that a slug of fluid oscillates within the matrix without exiting either end. The complexity of the problem and the ‘*natural*’ use of the Lagrange system have required (1) to classify the gas particles as ‘cold’, ‘hot’ and ‘internal’ particles according to their initial locations and, hence, (2) to deal separately with these three, independent, flow regimes separated by temperature discontinuities. In such a way, a ‘*segmented*’ solution of the regenerator problem is obtained. In particular, the cold and hot elements of gas come from the cold and hot spaces adjacent the regenerator, respectively. In the case of a Stirling machine, for instance, the cold slug of gas enters from the compression space, the hot one enters from the expansion and the internal oscillates without leaving the regenerative exchanger. The above concept of the independent solutions was first proposed by Organ in Ref. [28], where the thermal performance of isothermal heat exchangers of Stirling cycle machines was studied. A brief description of this notion is also given in his first book [11, chapter 2].

For that reason, two zones have been characterized within the matrix, the so-called ‘cold’ and ‘hot’ zones which may be reached by the cold and hot particles of fluid. For $1 < \alpha \leqslant 2$ there is a partial superposition of the above two zones that complicates the treatment in that the ‘superposition’ zone contains particles of any type. For $\alpha > 2$, instead, there is no superposition of the hot and cold zones and there exists a zone containing only internal elements of fluid. After proceeding to classify the particles of fluid and the zones of the matrix, the gas energy equation has analytically been integrated along the law of motion of any type of gas particle for both the blow and reverse periods and any value of the flush ratio. Then, following Organ’s approach [5,6], a linear temperature distribution for the matrix has been assumed. It has allowed the fluid temperatures as well as the regenerator effectiveness to be obtained in a closed-form. The obtained results have shown that the effectiveness is underestimated by the classical regenerator theory as it does not account for the effects of the gas remaining within the regenerator during the rapidly flow reversals (flush phase). Also, a comparison with numerical results of Organ’s model applied to metallic matrices (very high values of β) has shown an excellent agreement.

2. Mathematical formulation for cyclic operation in counterflow

The regenerator defining equations represent the transfer of heat to/from the working fluid and from/to the matrix on the passage of fluid through the regenerator. They may be derived from mass, momentum and energy balances concerning an element dx at a location x of the working fluid and matrix using an Euler formulation (Fig. 1a). Thus, for both transient and cyclic operation we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial p}{\partial x} + f_F \rho \frac{u^2}{2 r_h} = 0 \tag{2}$$

$$A_f \frac{\partial}{\partial t}(\rho c_v T) + A_f \frac{\partial}{\partial x}[\rho u c_p T] = hP(T_w - T) \tag{3}$$

$$-(A - A_f)\frac{\partial(\rho_w c_w T_w)}{\partial t} = hP(T_w - T) \tag{4}$$

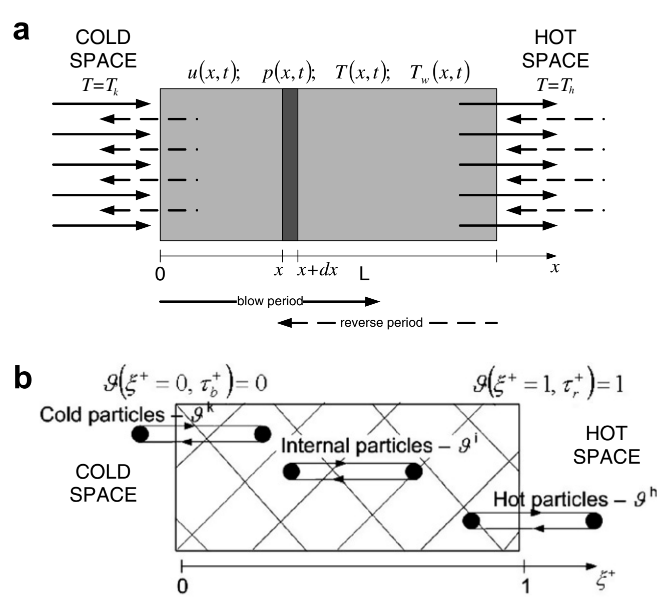

Fig. 1. Schematic representation for the analysis of the regenerator. (a) General scheme; (b) ‘cold’, ‘hot’ and ‘internal’ particles of gas.

where f_F is the Fanning friction factor and p the gas pressure. In the general case, in fact, the gas flowing through a regenerative matrix may be exposed to significant pressure and velocity fluctuations. Also, the temperature variation between the hot and the cold space can be relevant, leading to temperature-dependent fluid properties. The solution of Eqs. (1)–(4) may be tracked down only by a numerical approach, as done, for example, by Organ [18]. For that purpose, he considered an adequate integration grid in *natural coordinates* and he showed that

1. velocity fluctuations do not cause significant variations with respect to the constant-velocity case;
2. pressure fluctuations can induce a temperature swing for the gas at the matrix ends that may have a sensible influence on the regenerator effectiveness;
3. temperature-dependent properties can affect the gas thermal field which hence may show marked asymmetry in the form of convexity towards the higher temperature. However, as this is not close to the hot matrix end, the above dependence does not cause significant variations with respect to the constant-property case.

These findings would indicate that a simplified modelling of the regenerator problem may be formulated apart from the pressure fluctuations, which are negligible in some applications [7,8] but remain a subject for future research. In addition, effectiveness charts were given by Organ only in the simplified case of constant flow velocity and pressure and of gas properties independent of temperature. Also, no algebraic expression for the effectiveness as a function of the flush ratio is available in the regenerator literature.

Then, ignoring longitudinal heat conduction, pressure and velocity time-variations and assuming that the gas–matrix heat transfer coefficient is the same everywhere in the regenerator, the governing equations may be written for $x^+ \in [0,1]$ and $t_j^+ \in [t_{ij}^+, t_{ij}^+ + 1/2]$ in a dimensionless form as [26]

- gas energy balance

$$\frac{\alpha}{2}\frac{\partial \vartheta_j(x^+,t_j^+)}{\partial t_j^+} + \mathrm{sign}(u_j)\frac{\partial \vartheta_j(x^+,t_j^+)}{\partial x^+} = N_{\mathrm{TU}}[\vartheta_w(x^+,t_j^+) - \vartheta_j(x^+,t_j^+)] \quad (j = b, r) \tag{5}$$

- matrix energy balance

$$N_{\mathrm{TU}}[\vartheta_w(x^+,t_j^+) - \vartheta_j(x^+,t_j^+)] = -\frac{\alpha\beta}{2}\frac{\partial \vartheta_w(x^+,t_j^+)}{\partial t_j^+} \quad (j = b, r) \tag{6}$$

- cyclic steady operation (the heat released from the matrix during the flow of the cold gas stream – blow period – is equal at any location to the heat transferred to the matrix during the flow of the hot gas stream – reverse period)

$$\int_n^{n+1/2} [\vartheta_w(x^+,t_b^+) - \vartheta_b(x^+,t_b^+)]\mathrm{d}t_b^+ = \int_{n+1/2}^{n+1} [\vartheta_r(x^+,t_r^+) - \vartheta_w(x^+,t_r^+)]\mathrm{d}t_r^+ \tag{7}$$

- boundary conditions

$$\begin{aligned} &\vartheta_b(x^+ = 0, t_b^+) = 0 \quad \text{for } t_b^+ \in [n, n+1/2] \\ &\vartheta_r(x^+ = 1, t_r^+) = 1 \quad \text{for } t_r^+ \in [n+1/2, n+1] \end{aligned} \tag{8}$$

where
- 'sign(u_j)' appearing in Eq. (5) takes account of the possibility of positive and negative u_j as the flow direction alternates. Notice that the space coordinate system labelled in Fig. 1 considers as positive the gas flow from the cold end of the regenerator to the hot one, i.e. $u_b = -u_r = u > 0$;
- $x^+ = x/L$ is the dimensionless abscissa referred to the regenerator length L;
- $t^+ = t/\tau_0$ is the dimensionless time referred to the duration of a blow τ_0;
- $\vartheta(\vartheta_w) = \frac{T(T_w)-T_k}{T_h-T_k}$ is the non-dimensional temperature referred to the temperature jump between the hot and cold spaces.

For slowly switched heat regenerators ($\alpha \to 0$) using a metallic matrix ($\beta \to \infty \Rightarrow \alpha\beta = $ finite), the partial derivative $\partial\vartheta_j/\partial t_j^+$ on the left-hand side of Eq. (5) (which represents the heat storage in the gas) vanishes according to Hausen's theory [1]. For rapidly switched heat regenerators ($\alpha = $ finite) employing a metallic matrix ($\beta \to \infty$), which are here of interest, Eq. (2) reduces to $\partial\vartheta_w/\partial t_j^+ = 0$ as $\alpha\beta \to \infty$. This indicates that the matrix temperature at each location x^+ is essentially constant with time and hence the defining Eqs. (5) and (7) may be taken as

$$\frac{\alpha}{2}\frac{\partial \vartheta_j(x^+,t_j^+)}{\partial t_j^+} + \mathrm{sign}(u_j)\frac{\partial \vartheta_j(x^+,t_j^+)}{\partial x^+} = N_{\mathrm{TU}}[\vartheta_w(x^+) - \vartheta_j(x^+,t_j^+)] \quad (j = b, r) \tag{9}$$

$$\vartheta_w(x^+) = \int_n^{n+1/2} \vartheta_b(x^+,t_b^+)\mathrm{d}t_b^+ + \int_{n+1/2}^{n+1} \vartheta_r(x^+,t_r^+)\mathrm{d}t_r^+ \tag{10}$$

with the boundary conditions for the fluid temperature always given by Eq. (4). Thus, the mathematical formulation leads to the governing equations (8)–(10) whose unknowns are $\vartheta_j(x^+,t_j^+)$ ($j = b$ or r) and $\vartheta_w(x^+)$.

3. General solution of the gas energy equation

As was done in the early paper [26] and following Organ's approach [5,6,18] based on a Lagrange frame of reference, the term on the left-hand side of Eq. (9) is recognized to be the substantial derivative, $\mathrm{d}/\mathrm{d}t_j^+$, of the fluid temperature. Thus, Eq. (9) becomes

$$\frac{d}{dt_j^+}\vartheta_j[x_j^+(t_j^+),t_j^+]+\frac{2N_{TU}}{\alpha}\vartheta_j[x_j^+(t_j^+),t_j^+] = \frac{2N_{TU}}{\alpha}\vartheta_w[x_j^+(t_j^+)] \tag{11}$$

where $j = b, r$ and the law of motion of the gas particles in a dimensionless form is

$$x_j^+(t_j^+) = \xi^+ + \frac{2}{\alpha}\mathrm{sign}(u_j)(t_j^+ - \tau_j^+) \tag{12}$$

Notice that the ξ^+ denotes a generic location of an element of gas within the regenerator (i.e. $\xi^+ \in [0,1]$) to which corresponds a generic time τ_j^+. Now, it is interesting to observe that the integration of the linear first-order differential equation (11) along the path (12) followed by a selected gas particle may be performed analytically [29]. Its solution gives the temperature of an element of gas when it is positioned at the location ξ^+ and time τ_j^+, that is

$$\vartheta_j(\xi^+,\tau_j^+) = \vartheta_j(x_{ij}^+,t_{ij}^+)e^{-\frac{2N_{TU}}{\alpha}(\tau_j^+-t_{ij}^+)} + \frac{2N_{TU}}{\alpha}\int_{t_{ij}^+}^{\tau_j^+}\vartheta_w[x_j^+(t_j^+)]e^{\frac{2N_{TU}}{\alpha}(t_j^+-\tau_j^+)}\,dt_j^+ \tag{13}$$

where $x_{ij}^+ = x_j^+(t_{ij}^+)$ is the 'initial' location of a gas particle, that is, the location of a gas particle at the time t_{ij}^+ to which corresponds the beginning of the jth period (blow period or reverse period) of the nth operation cycle ($t_{ib}^+ = n$ and $t_{ir}^+ = 1/2 + n$). It may be evaluated through Eq. (12) simply setting $t_j^+ = t_{ij}^+$

$$x_{ij}^+ = \xi^+ + \frac{2}{\alpha}\mathrm{sign}(u_j)(t_{ij}^+ - \tau_j^+) \tag{14}$$

Now, by means of an algebraic substitution of the dummy variable in the integral on the RHS of Eq. (13), that is, time variable $t_j^+ \to$ space variable x^+ according to the law of motion (12), after some algebraic steps making use of Eq. (14), Eq. (13) becomes

$$\vartheta_j(\xi^+,\tau_j^+) = e^{-\frac{N_{TU}}{\mathrm{sign}(u_j)}\xi^+}\left[\vartheta_j(x_{ij}^+,t_{ij}^+)e^{\frac{N_{TU}}{\mathrm{sign}(u_j)}x_{ij}^+} + \frac{N_{TU}}{\mathrm{sign}(u_j)}\int_{x_{ij}^+}^{\xi^+}\vartheta_w(x^+)e^{\frac{N_{TU}}{\mathrm{sign}(u_j)}x^+}\,dx^+\right] \tag{15}$$

where x_{ij}^+ depends on the time τ_j^+ in the form of Eq. (14). Eq. (15) has the advantage to provide us the fluid temperatures in a much more straightforward manner than Eq. (13). Now, it follows from Eq. (15) that the computation of the fluid temperature requires

- the knowledge of matrix temperature ϑ_w. It may be evaluated through the cyclic operation condition (10) that, according to the Lagrange system here assumed, has to be rewritten as

$$\vartheta_w(\xi^+) = \int_n^{n+1/2}\vartheta_b(\xi^+,\tau_b^+)d\tau_b^+ + \int_{n+1/2}^{n+1}\vartheta_r(\xi^+,\tau_r^+)d\tau_r^+ \tag{16}$$

As ϑ_b and ϑ_r may be taken through Eq. (15) as a function of ϑ_w, Eq. (16) becomes an integral equation in the unknown ϑ_w. However, following Organ's approach [5,6,18] a uniform gradient for the matrix temperature has been assumed (Section 7). In particular, we have used the well-known expression deriving from the Nusselt classical theory of the regenerators ($\alpha \to 0$), i.e.

$$\vartheta_w(\xi^+) = \frac{N_{TU}}{N_{TU}+2}\xi^+ + \frac{1}{N_{TU}+2} \tag{17}$$

- the knowledge of 'initial' temperature $\vartheta_j(x_{ij}^+,t_{ij}^+) = \vartheta_{ij}$ of the gas particle, that is, the temperature of the gas particles when the blow and reverse periods start. It depends on the initial location of the element of gas. When the blow period starts, for example, the element of gas may be located inside or outside the regenerator. Similarly, when the reverse period starts. For this reason, the boundary conditions (8) are relevant only for the fluid particles located outside the regenerative matrix.

For $\alpha \leqslant 1$, the initial temperatures of the gas particles and subsequently the blow and reverse fluid temperatures have been derived in [26]. For $\alpha > 1$ (which is of interest in the current paper), the mathematical treatment is much more complex because a slug of fluid oscillates within the matrix without exiting either end. In such a case, to determine the initial temperatures of the gas particles, it is convenient to classify them as *cold* (k), *hot* (h) and *internal* (i) particles according to their initial locations, as shown in Fig. 1b. In particular, these locations may be linked to ξ^+ and τ_j^+ by using Eq. (14), where the ξ^+-range depends on the value of α and type of particle, as shown in Sections 4–6.

4. 'Cold' particles

During the cyclic operation of the regenerator, the 'cold' gas particles fluctuate between the cold space and regenerator without entering the hot space, as shown in Fig. 1.

4.1. Blow period

At the beginning of the blow period ($t_{ib}^+ = n$), the cold particles are located inside the cold space, i.e. $x_{ib}^+ \in [-(1/\alpha - \xi^+), 0]$. For $\alpha \leqslant 1$, the cold particles can reach any position ξ^+ within the regenerator, i.e. $\xi^+ \in [0,1]$. For $\alpha > 1$, instead, the cold particles of gas can reach only those positions having $\xi^+ \in [0, 1/\alpha]$ which define the so-called 'cold' zone of the matrix, as shown in Fig. 2. In both cases, however, an element of gas starting from the location $x_{ib}^+ \in [-(1/\alpha - \xi^+), 0]$ reaches the position ξ^+ at the time $\tau_b^+ \in [n + \xi^+\alpha/2, n + 1/2]$. In fact, substituting $x_{ib}^+ = -(1/\alpha - \xi^+)$ in Eq. (14) for $j = b$, we have $\tau_b^+ = n + 1/2$. Similarly, setting $x_{ib}^+ = 0$ in the same equation, we obtain $\tau_b^+ = n + \xi^+\alpha/2$.

Also, it is of great concern to observe that in the first part of the blow period ($t_b^+ \in [n, \tau_b^{k+}]$) there is no heat

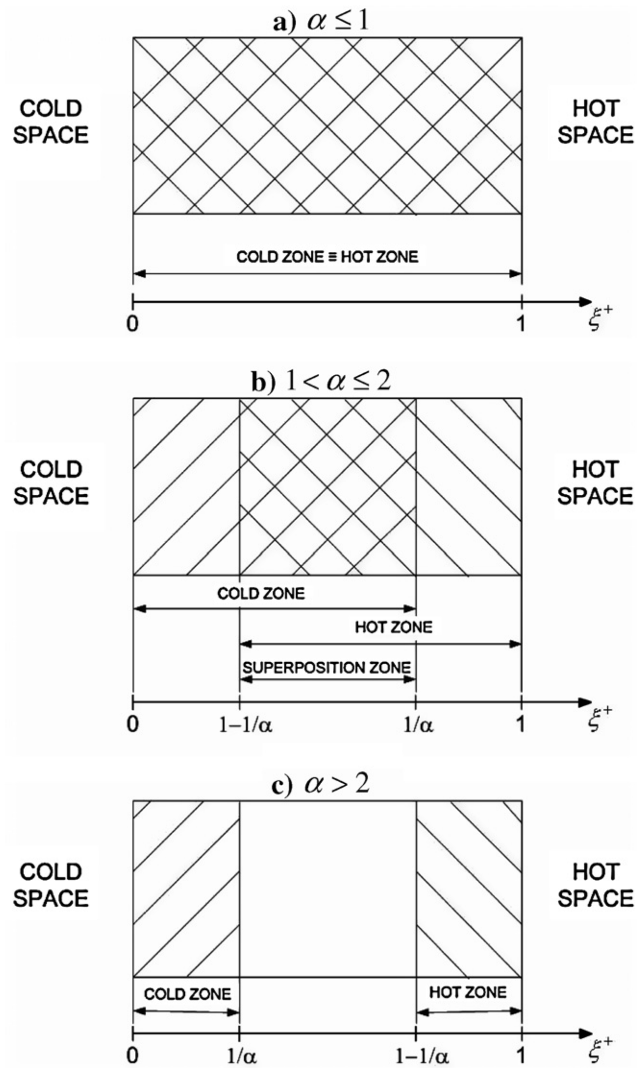

Fig. 2. 'Cold' and 'hot' zones of the matrix. (a) $\alpha \leqslant 1$ (complete superposition with no internal particles of gas); (b) $1 < \alpha \leqslant 2$ (partial superposition containing cold, hot and internal elements of gas); and (c) $\alpha > 2$ (zone $\xi^+ \in [1/\alpha, 1 - 1/\alpha]$ containing only the internal particles).

exchanged between the cold elements of gas and the matrix as these elements are outside the regenerator. (The time τ_b^{k+} may be evaluated through Eq. (12) simply setting $x_b^+(t_b^+) = 0$.) This would indicate that the initial time $t_{ib}^+ = n$ appearing in Eq. (15) for the blow period ($j = b$) has to be replaced by the 'heat exchange' initial time τ_b^{k+}. Similarly, the initial location x_{ib}^+ which appears in Eq. (15) for $j = b$ has to be replaced by the position $x_b^+(\tau_b^{k+}) = 0$, where the dimensionless temperature is equal to zero (see Eq. (8.1)). From what it has been said and bearing in mind Eq. (15), it follows that the temperature of a cold element of gas at the location ξ^+ and time τ_b^+ may be taken as

$$\vartheta_b^k(\xi^+) = N_{\mathrm{TU}}\, \mathrm{e}^{-N_{\mathrm{TU}}\xi^+} \int_0^{\xi^+} \vartheta_w(x^+)\mathrm{e}^{N_{\mathrm{TU}}x^+}\, \mathrm{d}x^+ \qquad (18)$$

Notice that the above temperature is independent of ξ^+.

4.2. Reverse period

At the beginning of the reverse period ($t_{ir}^+ = n + 1/2$), the cold particles are located inside the regenerator. For $\alpha \leqslant 1$, $x_{ir}^+ \in [\xi^+, 1]$ where $\xi^+ \in [0, 1]$. In this case, the gas particle reaches the position ξ^+ at the time $\tau_r^+ \in [n + 1/2, n + 1/2 + (1 - \xi^+)\alpha/2]$. In fact, setting $x_{ir}^+ = \xi^+$ in Eq. (14) for $j = r$, we have $\tau_r^+ = n + 1/2$. Similarly, substituting $x_{ir}^+ = 1$ in the same equation, we obtain the upper limit of the above time interval. For $\alpha > 1$, instead, $x_{ir}^+ \in [\xi^+, 1/\alpha]$ where $\xi^+ \in [0, 1/\alpha]$ ('cold' zone of the matrix). In fact, if this range was not verified, the cold particles would not enter the cold space and hence they could not be classified as cold particles. In such a case, the element of gas reaches the position ξ^+ at the time $\tau_r^+ \in [n + 1/2, n + 1 - \xi^+\alpha/2]$. In fact, setting $x_{ir}^+ = 1/\alpha$ in Eq. (14) for $j = r$, we have the upper limit of the above time interval. From what it has been stated and bearing in mind Eq. (15), the temperature of a cold element of fluid at the location ξ^+ and time τ_r^+ may be evaluated as

$$\vartheta_r^k(\xi^+, \tau_r^+) = \mathrm{e}^{N_{\mathrm{TU}}\xi^+}\left[\vartheta_r^k(x_{ir}^+, n + 1/2)\mathrm{e}^{-N_{\mathrm{TU}}x_{ir}^+} - N_{\mathrm{TU}} \int_{x_{ir}^+}^{\xi^+} \vartheta_w(x^+)\mathrm{e}^{-N_{\mathrm{TU}}x^+}\, \mathrm{d}x^+\right] \qquad (19)$$

where the initial temperature $\vartheta_r^k(x_{ir}^+, n + 1/2)$ depends on the thermal history of the same element during the preceding blow period ($t_b^+ \in [n, n + 1/2]$), that is, $\vartheta_r^k(x_{ir}^+, n + 1/2) = \vartheta_b^k(x_{ir}^+, n + 1/2)$. Now, the latter temperature may be obtained by means of Eq. (18) simply setting $\xi^+ = x_{ir}^+$. Thus, Eq. (19) becomes

$$\vartheta_r^k(\xi^+, \tau_r^+) = N_{\mathrm{TU}}\, \mathrm{e}^{N_{\mathrm{TU}}\xi^+}\left[\mathrm{e}^{-2N_{\mathrm{TU}}x_{ir}^+} \int_0^{x_{ir}^+} \vartheta_w(x^+)\mathrm{e}^{N_{\mathrm{TU}}x^+}\, \mathrm{d}x^+ - \int_{x_{ir}^+}^{\xi^+} \vartheta_w(x^+)\mathrm{e}^{-N_{\mathrm{TU}}x^+}\, \mathrm{d}x^+\right] \qquad (20)$$

where x_{ir}^+ depends on the time τ_r^+ in the form of Eq. (14) for $j = r$.

5. 'Hot' particles

When the regenerator works in cyclic operation, the 'hot' gas particles fluctuate between the regenerator and hot space without entering the cold space, as shown in Fig. 1.

5.1. Reverse period

At the beginning of the reverse period ($t_{ir}^+ = n + 1/2$), the hot particles are located inside the hot space, i.e. $x_{ir}^+ \in [1, 1/\alpha + \xi^+]$. For $\alpha \leqslant 1$, the hot particles can reach any position ξ^+ within the regenerator, i.e. $\xi^+ \in [0, 1]$. For $\alpha > 1$, instead, these particles of gas can reach only those positions having $\xi^+ \in [1 - 1/\alpha, 1]$ which define the so-called 'hot' zone of the matrix, as shown in Fig. 2. In both cases, however, a gas particle starting with the location $x^+_{ir} \in [1, 1/\alpha + \xi^+]$ reaches the position ξ^+ at $\tau^+_r \in [n + 1/2 + (1 - \xi^+)\alpha/2, n + 1]$. In fact, setting $x^+_{ir} = 1$ in Eq. (14) for $j = r$, we have the lower limit of the above time interval. Similarly, substituting $x^+_{ir} = 1/\alpha + \xi^+$ in the same equation, we obtain $\tau^+_r = n + 1$.

Also, it is relevant to note that in the first part of the reverse period ($t^+_r \in [n + 1/2, \tau^{h+}_r]$) there is no heat transferred between the gas particles and matrix as these particles are outside the regenerator. (The time τ^{h+}_r may be evaluated through Eq. (12) simply setting $x^+_r(t^+_r) = 1$.) This indicates that the initial time $t^+_{ir} = n + 1/2$ appearing in Eq. (15) for the reverse period ($j = r$) has to be replaced by the 'heat transfer' initial time τ^{h+}_r. Similarly, the initial location x^+_{ir} has to be replaced by the position $x^+_r(t^+_r) = 1$, where the dimensionless temperature is equal to unity (see Eq. (8.2)). Thus, applying Eq. (15) gives the temperature of a hot element of gas at the location ξ^+ and time τ^+_r as

$$\vartheta^h_r(\xi^+) = \mathrm{e}^{N_{\mathrm{TU}}\xi^+}\left[\mathrm{e}^{-N_{\mathrm{TU}}} - N_{\mathrm{TU}}\int_1^{\xi^+}\vartheta_w(x^+)\mathrm{e}^{-N_{\mathrm{TU}}x^+}\,\mathrm{d}x^+\right] \tag{21}$$

Notice that the above temperature is independent of ξ^+.

5.2. Blow period

At the beginning of the blow period ($t^+_{ib} = n$), the hot particles are located inside the regenerator. For $\alpha \leqslant 1$, $x^+_{ib} \in [0, \xi^+]$ where $\xi^+ \in [0,1]$. In this case, the gas particle reaches the position ξ^+ at the time $\tau^+_b \in [n, n + \xi^+\alpha/2]$. In fact, setting $x^+_{ib} = 0$ in Eq. (14) for $j = b$, we have $\tau^+_b = n + \xi^+\alpha/2$. Similarly, substituting $x^+_{ib} = \xi^+$ in the same equation, we get $\tau^+_b = n$. For $\alpha > 1$, instead, $x^+_{ib} \in [1 - 1/\alpha, \xi^+]$ where $\xi^+ \in [1 - 1/\alpha, 1]$. In fact, if this range was not verified, the hot particles would not enter the hot space and hence they could not be classified as hot particles. In such a case, the element of gas reaches the position ξ^+ at the time $\tau^+_b \in [n, n + (1 - \alpha)/2 + \xi^+\alpha/2]$. In fact, setting $x^+_{ib} = 1 - 1/\alpha$ in Eq. (14) for $j = b$, we have the upper limit of the above time interval. From what it has been stated and applying Eq. (15), the temperature of a hot fluid element at the location ξ^+ and time τ^+_b may be evaluated as

$$\vartheta^h_b(\xi^+, \tau^+_b) = \mathrm{e}^{-N_{\mathrm{TU}}\xi^+}\left[\vartheta^h_b(x^+_{ib}, n)\mathrm{e}^{N_{\mathrm{TU}}x^+_{ib}} + N_{\mathrm{TU}}\int_{x^+_{ib}}^{\xi^+}\vartheta_w(x^+)\mathrm{e}^{N_{\mathrm{TU}}x^+}\,\mathrm{d}x^+\right] \tag{22}$$

where the initial temperature $\vartheta^h_b(x^+_{ib}, n)$ depends on the thermal history of the same element during the reverse period at the $(n - 1)$th operation cycle ($t^+_r \in [n - 1/2, n]$), that is, $\vartheta^h_b(x^+_{ib}, n) = \vartheta^h_r(x^+_{ib}, n)$. Now, the latter temperature may be obtained by means of Eq. (21) simply setting $\xi^+ = x^+_{ib}$. Thus, Eq. (22) becomes

$$\vartheta^h_b(\xi^+, \tau^+_b) = \mathrm{e}^{-N_{\mathrm{TU}}\xi^+}\left[\mathrm{e}^{N_{\mathrm{TU}}(2x^+_{ib}-1)} - N_{\mathrm{TU}}\,\mathrm{e}^{2N_{\mathrm{TU}}x^+_{ib}}\int_1^{x^+_{ib}}\vartheta_w(x^+)\mathrm{e}^{-N_{\mathrm{TU}}x^+}\,\mathrm{d}x^+ + N_{\mathrm{TU}}\int_{x^+_{ib}}^{\xi^+}\vartheta_w(x^+)\mathrm{e}^{N_{\mathrm{TU}}x^+}\,\mathrm{d}x^+\right] \tag{23}$$

where x^+_{ib} depends on the time τ^+_b in the form of Eq. (14) for $j = b$.

6. 'Internal' particles

During the cyclic operation of the regenerator, the 'internal' particles of gas oscillate within the matrix without exiting either end, as shown in Fig. 1. Of course, they vanish for $\alpha \leqslant 1$ and we would have only the cold and hot particles treated in the previous two sections.

6.1. Blow period

At the beginning of the blow period ($t^+_{ib} = n$), the internal particles are located at $x^+_{ib} \in [0, 1 - 1/\alpha]$. In fact, if this range was not verified, the internal particles would enter the hot space and hence they could not be classified as internal particles. Thus, we have:

- $1 < \alpha \leqslant 2$. In this case, the internal particles, starting from $x^+_{ib} \in [0, 1 - 1/\alpha]$, can reach both the domains $\xi^+ \in [0, 1 - 1/\alpha]$ and $\xi^+ \in [1 - 1/\alpha, 1/\alpha]$ where the latter is given by the superposition of the cold ($[0, 1/\alpha]$) and hot ($[1 - 1/\alpha, 1]$) zones of the matrix (see Fig. 2b). Notice that an internal particle starting from $x^+_{ib} \in [\xi^+ - 1/\alpha, 1 - 1/\alpha]$ can also reach the domain $\xi^+ \in [1/\alpha, 1]$.
- $\alpha > 2$. In the current case, the internal particles, starting from $x^+_{ib} \in [0, 1 - 1/\alpha]$, can reach both the $\xi^+ \in [0, 1/\alpha]$ and $\xi^+ \in [1/\alpha, 1 - 1/\alpha]$ domains where the former is the cold zone of the matrix (Fig. 2c). Also, an internal particle starting from $x^+_{ib} \in [\xi - 1/\alpha, 1 - 1/\alpha]$ can reach the domain $\xi^+ \in [1 - 1/\alpha, 1]$ (hot zone of the matrix).

Applying Eq. (15) for $j = b$, the temperature of an internal element of fluid at the location ξ^+ and time τ^+_b may be evaluated as

$$\vartheta^i_b(\xi^+, \tau^+_b) = \vartheta^i_b(x^+_{ib}, n)\mathrm{e}^{N_{\mathrm{TU}}(x^+_{ib}-\xi^+)} + N_{\mathrm{TU}}\int_{x^+_{ib}}^{\xi^+}\vartheta_w(x^+)\mathrm{e}^{N_{\mathrm{TU}}(x^+-\xi^+)}\,\mathrm{d}x^+ \tag{24}$$

where the initial temperature $\vartheta^i_b(x^+_{ib}, n)$ may be calculated making use of the cyclic operation of the regenerator, that is

$$\vartheta^i_b(x^+_{ib}, n) = \vartheta^i_b(x^+_{ib}, n - 1) \tag{25}$$

Now, the temperature on the LHS of the above equation depends on the thermal history of the same particle during the reverse period of the $(n-1)$th operation cycle $(t_r^+ \in [n-1/2, n])$, i.e. $\vartheta_b^i(x_{ib}^+, n) = \vartheta_r^i(\xi^+ = x_{ib}^+, \tau_r^+ = n)$. In particular, the latter temperature may be derived applying Eq. (15) for $j = r$ and $t_{ir}^+ = n - 1/2$. Also, it depends on $\vartheta_r^i(x_{ir}^+, n-1/2)$ which is the temperature of the internal particles at the beginning of the $(n-1)$th reverse period with $x_{ir}^+ = x_{ib}^+ + 1/\alpha$. However, since this temperature may also be seen as the temperature at the end of the $(n-1)$th blow period $(t_b^+ \in [n-1, n-1/2])$, we can write $\vartheta_r^i(x_{ir}^+, n-1/2) = \vartheta_b^i(\xi^+ = x_{ir}^+, \tau_b^+ = n-1/2)$ where the second temperature may be determined by means of Eq. (15) for $j = b$ and $t_{ib}^+ = n-1$. Notice also that it depends on $\vartheta_b^i(x_{ib}^+, n-1)$ which is the temperature of the internal particles at the beginning of the $(n-1)$th blow period, that is, the temperature on the RHS of Eq. (25). Thus, solving Eq. (25) for $\vartheta_b^i(x_{ib}^+, n)$, we obtain

$$\vartheta_b^i(x_{ib}^+, n) = \frac{N_{\mathrm{TU}}}{1 - \mathrm{e}^{-2\frac{N_{\mathrm{TU}}}{\alpha}}} \left\{ \int_{x_{ib}^+}^{x_{ib}^+ + \frac{1}{\alpha}} \vartheta_w(x^+) \left[\mathrm{e}^{-N_{\mathrm{TU}}\left(x_{ib}^+ + \frac{2}{\alpha} - x^+\right)} + \mathrm{e}^{N_{\mathrm{TU}}(x_{ib}^+ - x^+)} \right] \mathrm{d}x^+ \right\} \tag{26}$$

where x_{ib}^+ depends on the time τ_b^+ in the form of Eq. (14) for $j = b$. It is relevant to note that the time and space intervals where Eq. (24), along with the companion Eq. (26), may be used depend strictly on the value of α, as it will be illustrated in Section 7.

6.2. Reverse period

At the beginning of the reverse period $(t_{ir}^+ = n + 1/2)$, the internal particles are located at $x_{ir}^+ \in [1/\alpha, 1]$. In fact, if this range was not verified, the internal particles would enter the cold space and hence they could not be classified as internal particles. Therefore, we have:

- $1 < \alpha \leqslant 2$. In this case, the internal particles starting from $x_{ir}^+ \in [1/\alpha, 1]$ can reach both the domains $\xi^+ \in [1/\alpha, 1]$ and $\xi^+ \in [1 - 1/\alpha, 1/\alpha]$ where the latter is given by the superposition of the cold ($[0, 1/\alpha]$) and hot ($[1 - 1/\alpha, 1]$) zones of the matrix (see Fig. 2b). Notice that an internal particle starting from $x_{ir}^+ \in [1/\alpha + \xi^+, 1]$ can also reach the domain $\xi^+ \in [0, 1 - 1/\alpha]$.
- $\alpha > 2$. In the present case, the internal particles are still located initially at $x_{ir}^+ \in [1/\alpha, 1]$ and they can reach both the domains $\xi^+ \in [1 - 1/\alpha, 1]$ and $\xi^+ \in [1/\alpha, 1 - 1/\alpha]$ where the former is the hot zone of the matrix (Fig. 2c). Also, an internal particle starting from $x_{ir}^+ \in [1/\alpha + \xi^+, 1]$ can reach the domain $\xi^+ \in [0, 1/\alpha]$ (the cold zone of the matrix).

Applying Eq. (15) for $j = r$, the temperature of an internal element of fluid at the location ξ^+ and time τ_r^+ may be evaluated as

$$\vartheta_r^i(\xi^+, \tau_r^+) = \vartheta_r^i(x_{ir}^+, n + 1/2)\mathrm{e}^{-N_{\mathrm{TU}}(x_{ir}^+ - \xi^+)} - N_{\mathrm{TU}} \times \int_{x_{ir}^+}^{\xi^+} \vartheta_w(x^+)\mathrm{e}^{N_{\mathrm{TU}}(\xi^+ - x^+)}\,\mathrm{d}x^+ \tag{27}$$

where the initial temperature $\vartheta_r^i(x_{ir}^+, n + 1/2)$ may be calculated making use of the cyclic operation of the regenerator, that is, $\vartheta_r^i(x_{ir}^+, n + 1/2) = \vartheta_r^i(x_{ir}^+, n - 1/2)$. Then, following a procedure similar to the one applied to the blow period in the previous subsection and solving the above equation for $\vartheta_r^i(x_{ir}^+, n + 1/2)$, we have

$$\vartheta_r^i(x_{ir}^+, n + 1/2) = \frac{N_{\mathrm{TU}}}{1 - \mathrm{e}^{-2\frac{N_{\mathrm{TU}}}{\alpha}}} \left\{ \int_{x_{ir}^+ - \frac{1}{\alpha}}^{x_{ir}^+} \vartheta_w(x^+) \left[\mathrm{e}^{N_{\mathrm{TU}}\left(x_{ir}^+ - \frac{2}{\alpha} - x^+\right)} + \mathrm{e}^{N_{\mathrm{TU}}(x^+ - x_{ir}^+)} \right] \mathrm{d}x^+ \right\} \tag{28}$$

where x_{ir}^+ depends on the time τ_r^+ in the form of Eq. (14) for $j = r$. Notice that the time and space intervals where Eq. (27), along with the companion Eq. (28), may be used depend strictly on the value of α, as it will be described in Section 7.

7. Fluid temperature solution

As the matrix temperature $\vartheta_w(\xi^+)$ may be evaluated approximately through Eq. (17), Eqs. (18), (23) and (24) concerning blow period allow $\vartheta_b^k(\xi^+)$, $\vartheta_b^h(\xi^+, \tau_b^+)$ and $\vartheta_b^i(\xi^+, \tau_b^+)$ to be definitely evaluated. Similarly, Eqs. (20), (21) and (27) regarding reverse period allow $\vartheta_r^k(\xi^+, \tau_b^+)$, $\vartheta_r^h(\xi^+)$ and $\vartheta_r^i(\xi^+, \tau_r^+)$ to be calculated as well. Thus, solving the integrals on the RHS of the equations stated before, we have

$$\vartheta_j^p(\xi^+, \tau_j^+) = \frac{N_{\mathrm{TU}}}{N_{\mathrm{TU}} + 2}\xi^+ + \frac{1 - 1 \cdot \mathrm{sign}(u_j)}{N_{\mathrm{TU}} + 2} + \frac{2\theta_j^p\,\mathrm{sign}(u_j)}{N_{\mathrm{TU}} + 2}\mathrm{e}^{\frac{N_{\mathrm{TU}}}{\mathrm{sign}(u_j)}(x_{ij}^+ - \xi^+)} \quad (j = b, r \text{ and } p = h, i, k) \tag{29}$$

where $\theta_b^k = \theta_r^h = 0$, $\theta_b^h = \theta_r^k = 1$ and $\theta_b^i = \theta_r^i = [1 + \exp(-N_{\mathrm{TU}}/\alpha)]^{-1}$. (Notice the exponential dependence of the fluid temperature on the time.) The time and space intervals (including the values of α) where Eq. (29) may be used are summarised in Table 1. A flow chart illustrating the different calculation procedures named in this table is given in Fig. 3 for both the blow (a) and reverse (b) periods. For any space ξ^+ and time τ_j^+ location, it enables the type of gas particle (h, k for $\alpha \leqslant 1$, and k, h, i for $\alpha > 1$) within the regenerator to be established in order to appropriately calculate its initial location x_{ij}^+ and, hence, the fluid temperature.

Fig. 4 shows the fluid temperature for $N_{\mathrm{TU}} = 1$ and $\alpha = 5/3$ as a function of ξ^+ with τ_j^+ $(j = b, r)$ as a parameter for both the blow (a) and reverse (b) periods. The matrix temperature (time-independent) is plotted in the same diagrams and its cold and hot zones depending on α are

Table 1
Cyclic operation of the regenerator. Time and space intervals for the fluid temperatures as a function of α

Blow period ($\tau_b^+ \in [n, n+1/2]$)				Reverse period ($\tau_r^+ \in [n+1/2, n+1]$)			
α	ξ^+	τ_b^+	ϑ_b	α	ξ^+	τ_r^+	ϑ_r
$\alpha \leqslant 1$	$[0,1]$	$[n, \tau_{b,2}^+]$	$\vartheta_b^h(\xi^+, \tau_b^+)$	$\alpha \leqslant 1$	$[0,1]$	$[n+1/2, \tau_{r,2}^+]$	$\vartheta_r^k(\xi^+, \tau_r^+)$
		$[\tau_{b,2}^+, n+1/2]$	$\vartheta_b^k(\xi^+)$			$[\tau_{r,2}^+, n+1]$	$\vartheta_r^h(\xi^+)$
$1 < \alpha \leqslant 2$	$[0, 1-1/\alpha]$	$[n, \tau_{b,2}^+]$	$\vartheta_b^i(\xi^+, \tau_b^+)$	$1 < \alpha \leqslant 2$	$[0, 1-1/\alpha]$	$[n+1/2, \tau_{r,1}^+]$	$\vartheta_r^k(\xi^+, \tau_r^+)$
		$[\tau_{b,2}^+, n+1/2]$	$\vartheta_b^k(\xi^+)$			$[\tau_{r,1}^+, n+1]$	$\vartheta_r^i(\xi^+, \tau_r^+)$
	$[1-1/\alpha, 1/\alpha]$	$[n, \tau_{b,1}^+]$	$\vartheta_b^h(\xi^+, \tau_b^+)$		$[1-1/\alpha, 1/\alpha]$	$[n+1/2, \tau_{r,1}^+]$	$\vartheta_r^k(\xi^+, \tau_r^+)$
		$[\tau_{b,1}^+, \tau_{b,2}^+]$	$\vartheta_b^i(\xi^+, \tau_b^+)$			$[\tau_{r,1}^+, \tau_{r,2}^+]$	$\vartheta_r^i(\xi^+, \tau_r^+)$
		$[\tau_{b,2}^+, n+1/2]$	$\vartheta_b^k(\xi^+)$			$[\tau_{r,2}^+, n+1]$	$\vartheta_r^h(\xi^+)$
	$[1/\alpha, 1]$	$[n, \tau_{b,1}^+]$	$\vartheta_b^h(\xi^+, \tau_b^+)$		$[1/\alpha, 1]$	$[n+1/2, \tau_{r,2}^+]$	$\vartheta_r^i(\xi^+, \tau_r^+)$
		$[\tau_{b,1}^+, n+1/2]$	$\vartheta_b^i(\xi^+, \tau_b^+)$			$[\tau_{r,2}^+, n+1]$	$\vartheta_r^h(\xi^+)$
$\alpha > 2$	$[0, 1/\alpha]$	$[n, \tau_{b,2}^+]$	$\vartheta_b^i(\xi^+, \tau_b^+)$	$\alpha > 2$	$[0, 1/\alpha]$	$[n+1/2, \tau_{r,1}^+]$	$\vartheta_r^k(\xi^+, \tau_r^+)$
		$[\tau_{b,2}^+, n+1/2]$	$\vartheta_b^k(\xi^+)$			$[\tau_{r,1}^+, n+1]$	$\vartheta_r^i(\xi^+, \tau_r^+)$
	$[1/\alpha, 1-1/\alpha]$	$[n, n+1/2]$	$\vartheta_b^i(\xi^+, \tau_b^+)$		$[1/\alpha, 1-1/\alpha]$	$[n+1/2, n+1]$	$\vartheta_r^i(\xi^+, \tau_r^+)$
	$[1-1/\alpha, 1]$	$[n, \tau_{b,1}^+]$	$\vartheta_b^h(\xi^+, \tau_b^+)$		$[1-1/\alpha, 1]$	$[n+1/2, \tau_{r,2}^+]$	$\vartheta_r^i(\xi^+, \tau_r^+)$
		$[\tau_{b,1}^+, n+1/2]$	$\vartheta_b^i(\xi^+, \tau_b^+)$			$[\tau_{r,2}^+, n+1]$	$\vartheta_r^h(\xi^+)$
$\tau_{b,1}^+ = n + (1-\alpha)/2 + \xi^+\alpha/2$		$\tau_{b,2}^+ = n + \xi^+\alpha/2$		$\tau_{r,1}^+ = n + 1 - \xi^+\alpha/2$		$\tau_{r,2}^+ = n + 1/2 + (1-\xi^+)\alpha/2$	

pointed out. Notice that, when the gas flows from the cold space to the hot one (blow period), the fluid temperature is not continuous at the points (ξ^+, τ_b^+) which satisfy the times $\tau_{b,1}^+$ and $\tau_{b,2}^+$ given in Table 1. For example, when $\alpha = 5/3$ and $\tau_b^+ = n + 1/6$ (second plot in Fig. 4a), the gas temperature is discontinuous at $\xi^+ = 0.2$ and $\xi^+ = 0.6$, and so on. This discontinuity is due to the proposed analytical treatment which deals with the boundary condition (8.2). In fact, in order that this boundary condition can really occur, the dimensionless temperature of the gas exiting the matrix at its right-hand side ($x^+ = 1$) during the blow period ($t_b^+ \in [n, n+1/2]$) and entering the hot space has to approach suddenly the unity. The sudden variation of temperature causes the above discontinuity. Similarly, during the reverse period, the fluid temperature is not continuous at the points (ξ^+, τ_r^+) which satisfy the times $\tau_{r,1}^+$ and $\tau_{r,2}^+$ given again in Table 1. The cyclic propagation of temperature discontinuities can more revealingly be portrayed in contour plots and pseudo-three-dimensional temperature reliefs, as it is done in Fig. 5 for $N_{\mathrm{TU}} = 1$ and $\alpha = 10/3$. In this case, when $\tau_r^+ = 5/6$ at the nth cycle, the gas temperature is discontinuous at $\xi^+ = 0.1$ and $\xi^+ = 0.8$. This discontinuity is due to the boundary condition (8.1). In fact, in order that this boundary condition can really occur, the dimensionless temperature of the gas exiting the matrix at its left-hand side ($x^+ = 0$) during the reverse period ($t_r^+ \in [n+1/2, n+1]$) and entering the cold space has to become abruptly equal to zero. This abrupt variation causes the above temperature discontinuity. These discontinuities are in agreement with Organ's findings. In fact, using the simplification of incompressibility, Organ [28] demonstrated that one-dimensional, cyclically reversing flow in a duct is inevitably accompanied by discontinuities in the lengthwise distribution of temperature.

Fig. 4 also shows that, at a prescribed time τ_b^+ during the blow period, the matrix does in general not give up heat to the gas at all the locations ξ^+ within the regenerator. As an example, when $\alpha = 5/3$ and $\tau_b^+ = n + 1/6$ (second plot in Fig. 4a), the matrix temperature is higher than the fluid temperature for only $\xi^+ \in [0, 0.2]$. This indicates that the heat is transferred from the fluid to the matrix for $\xi^+ \in [0.2, 1]$ at the time $\tau_b^+ = n + 1/6$ even though the gas at that time is flowing from the cold space to the hot one. All this is due to the phenomenon of the flush phase occurring only in rapidly switched heat regenerators. Similar considerations may be done for the reverse period.

8. Effectiveness and heat stored in the regenerator

The regenerator effectiveness ε may be defined as the ratio of the heat actually exchanged to an ideal amount of heat which would be exchanged if the temperature of the cold gas could be increased to the entrance temperature of the hot gas [2, chapter 35]. Therefore, we have

$$\varepsilon = \frac{\overline{T}_b(\xi = L) - T_k}{T_h - T_k} = \bar{\vartheta}_b(\xi^+ = 1) \tag{30}$$

where $\bar{\vartheta}_b(\xi^+ = 1)$ is the time-average dimensionless temperature of the fluid over the blow period at the hot end of the regenerator ($\xi^+ = 1$). Bearing in mind the time and space intervals of Table 1, Eq. (30) has to be split into two cases. Thus, we have

$$\varepsilon = 2 \cdot \begin{cases} \int_n^{n+\alpha/2} \vartheta_b^h(\xi^+ = 1, \tau_b^+)\mathrm{d}\tau_b^+ \\ \quad + \int_{n+\alpha/2}^{n+1/2} \vartheta_b^k(\xi^+ = 1)\mathrm{d}\tau_b^+, & \alpha \leqslant 1 \\ \int_n^{n+1/2} \vartheta_b^h(\xi^+ = 1, \tau_b^+)\mathrm{d}\tau_b^+, & \alpha > 1 \end{cases} \tag{31}$$

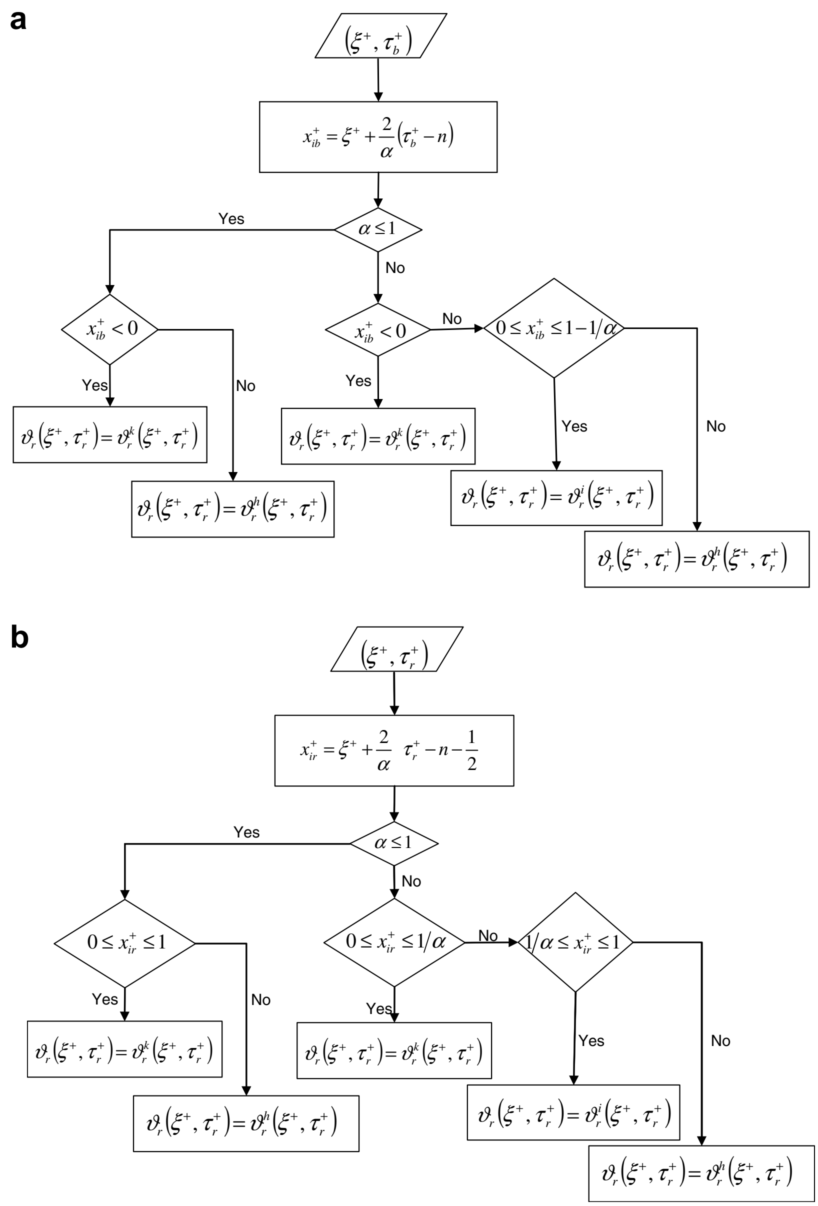

Fig. 3. Flow chart for the fluid temperature calculation. (a) Blow period; (b) reverse period.

Substituting Eq. (29) in Eq. (31), we obtain the effectiveness in an exact closed-form as a function of the flush ratio α and the number of transfer units N_{TU}, that is

$$\varepsilon(N_{TU},\alpha)=\frac{N_{TU}}{N_{TU}+2}+\frac{2\alpha}{N_{TU}(N_{TU}+2)}\cdot\begin{cases}(1-e^{-N_{TU}}), & \alpha\leqslant 1\\ (1-e^{-N_{TU}/\alpha}), & \alpha>1\end{cases} \tag{32}$$

The first term on the RHS of Eq. (32) is the well-known effectiveness $\varepsilon_{\alpha=0}$ deriving from the classical regenerator theory (Nusselt) and applicable only to slowly switched heat regenerators in countercurrent. Eq. (32) says that ε increases linearly with α for $\alpha\leqslant 1$. Eq. (32) also states that for $\alpha>1$ the effectiveness increases with α but less rapidly than a linear trend. In fact, for $\alpha\to\infty$ it approaches the unity whichever value is given to N_{TU}. In such a limiting case, however, only internal gas particles are present within the regenerator and, consequently, it does not work. Fig. 6 shows the regenerator effectiveness as a function of both α and N_{TU}. The effect of flush ratio on ε is considerable when the number of transfer units is low. This effect however decreases when N_{TU} increases. It may also be noted that the effectiveness is a monotonically increasing function with N_{TU} for low values of α. Contrary, for values of α approximately greater than 0.7, the effectiveness presents a trade-off in N_{TU} (in particular, a minimum value). For the limiting case $\alpha=1$, the mass of gas contained in the regenerator is equal to that passing through it in one blow period.

Once the efficiency is known, the so-called ‘regenerator losses’ due to imperfect heat transfer from gas to matrix and vice versa may be taken as $(1-\varepsilon)Q_r^+$, where Q_r^+ is the dimensionless heat stored in the regenerator during each blow. It is given by

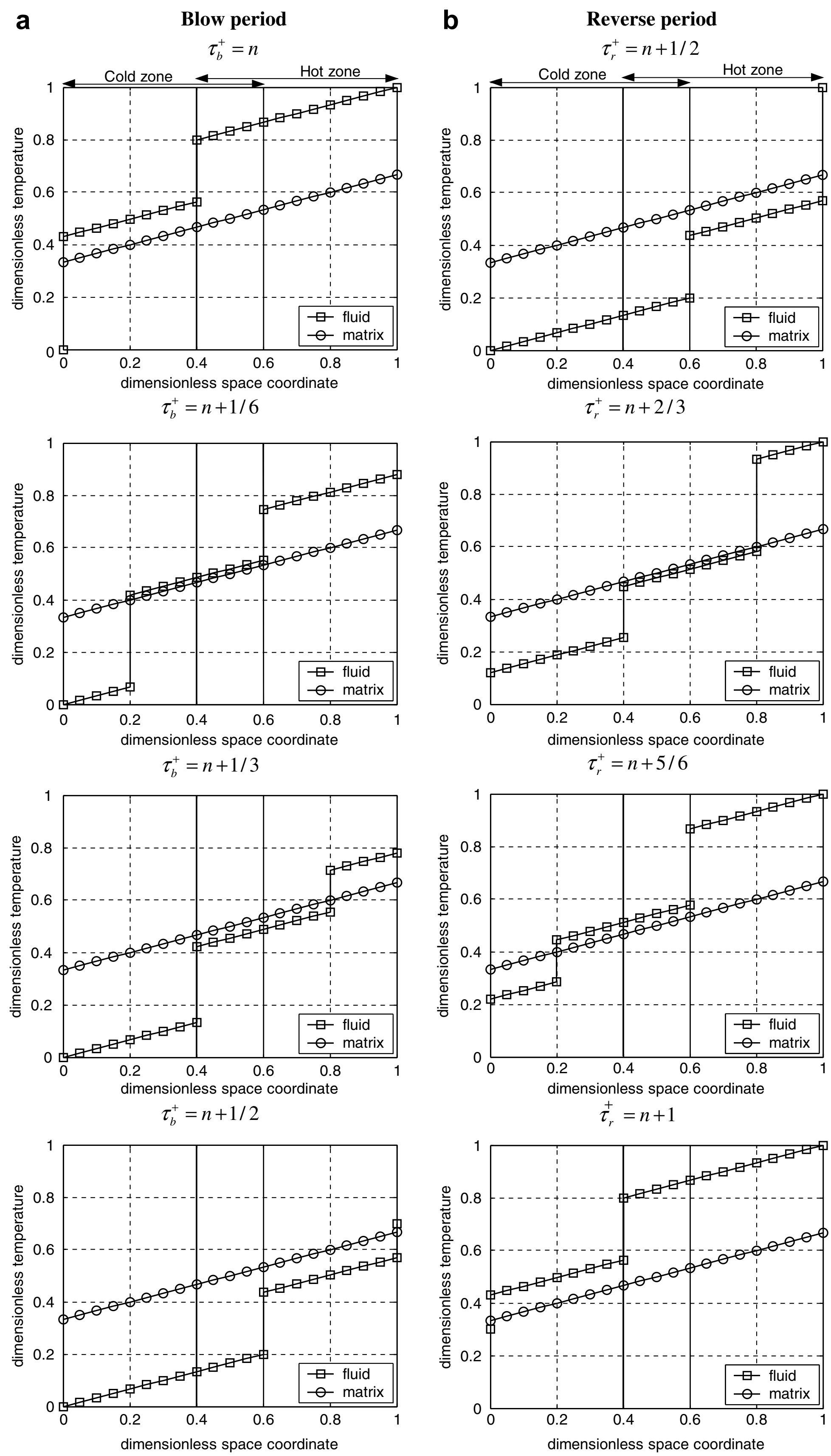

Fig. 4. Dimensionless temperatures (fluid and matrix) as a function of ξ^+ with τ_j^+ $(j = b, r)$ as a parameter for $N_{\mathrm{TU}} = 1$ and $\alpha = 5/3$. (a) Blow period; and (b) reverse period.

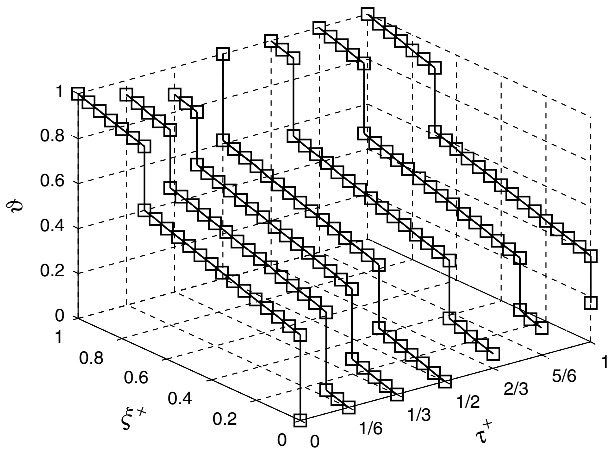

Fig. 5. Fluid dimensionless temperature versus ξ^+ and τ^+ for a whole operation cycle with $N_{TU} = 1$ and $\alpha = 10/3$. (Blow period when $\tau^+ \in [0, 1/2]$; reverse period when $\tau^+ \in [1/2, 1]$.)

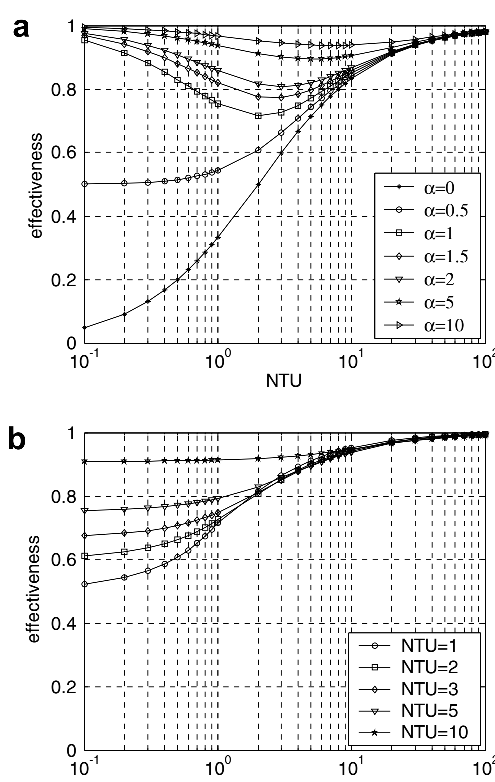

Fig. 6. Regenerator effectiveness (a) as a function of N_{TU} with α as a parameter, and (b) versus α with N_{TU} as a parameter.

$$Q_r^+ = \frac{\int_0^L \int_{n\tau_0}^{(n+1/2)\tau_0} hP(T_w - T_b)\,\mathrm{d}\tau_b\,\mathrm{d}\xi}{h(PL)(T_h - T_k)\tau_0} = \int_0^1 \int_n^{n+1/2} (\vartheta_w - \vartheta_b)\,\mathrm{d}\tau_b^+\,\mathrm{d}\xi^+ \tag{33}$$

where $PL = V/r_h$ is the matrix wetted area. Solving the above integral and using Eq. (32) give

$$Q_r^+ = \begin{cases} \frac{\varepsilon}{2N_{TU}} - \frac{\alpha}{N_{TU}(N_{TU}+2)}, & \alpha \leqslant 1 \\ \frac{\varepsilon}{2}\left(\frac{1}{N_{TU}} - \frac{1}{1+e^{-\frac{N_{TU}}{\alpha}}}\right) - \frac{4 - N_{TU}^2 e^{\frac{N_{TU}}{\alpha}}}{2N_{TU}(N_{TU}+2)\left(1+e^{\frac{N_{TU}}{\alpha}}\right)}, & \alpha > 1 \end{cases} \tag{34}$$

A comparison with numerical results of Organ's model [18, chapter 7], applied to metallic matrices ($N_{TCR} = 10^3$ and $N_{TCR} = 10^4$ where $N_{TCR} = \beta\phi/(1-\phi)$ with $\phi = A_f/A$ denoting the volumetric porosity) is given in Fig. 7 for $\alpha = 1/4$ (a) and $\alpha = 1/2$ (b). It shows an excellent agreement between the two procedures, in particular when $\alpha = 1/4$ (Fig. 7a). A certain discrepancy in the values of effectiveness occurs for $\alpha = 1/2$ (Fig. 7b) at small values of N_{TU} (around 1), since the non-linearities of matrix temperature in the vicinity of regenerator inlet and outlet as well as its cyclic temperature fluctuations are beyond the scope of the present treatment. In this case, the percent deviation is of about 10%. Notice, however, that in Stirling regenerator applications N_{TU} is usually greater than 20. For example, in the Philips MP1002CA air engine (250 W) with $T_k = 60$ °C, $T_h = 700$ °C and $PL = 0.455$ m^2 [18, p. 565], we have $N_{TU} = 37$ and $\alpha = 0.3$ which give $\varepsilon = 0.949$ and $Q_r^+ = 0.0126$ ($\Rightarrow$ a regenerator loss of about 37 W with $h = 100$ W/(m^2 K)). In the United Stirling P-40 'U' engine (4.5 kW) where $T_k = 60$ °C, $T_h = 750$ °C and $PL = 4.1$ m^2

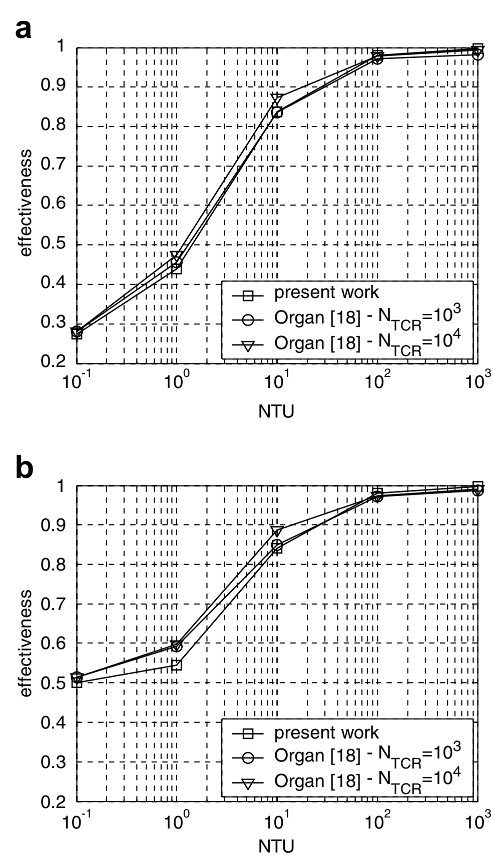

Fig. 7. Comparison with numerical results of Organ's model for (a) $\alpha = 1/4$ and (b) $\alpha = 1/2$.

[18, p. 569], we have $N_{TU} = 130$ and $\alpha = 1.45$ which provide an efficiency of about 0.985 and $Q_r^+ = 0.0037$ ($\Rightarrow$ a regenerator loss of 310 W).

It follows from the above comparison that Eqs. (32) and (34) may in every respect be used in an initial stage of the regenerator design, without the need of numerical computations. On the other hand, it must be pointed out that high temperature recovery cannot be the sole design criterion, as its increase with increasing N_{TU} goes always with a corresponding increase in pumping losses along the regenerator. The optimum N_{TU} value should not therefore be as high as possible (to maximize ε), but the best compromise between thermal performance and pumping losses, as indicated by Organ [5,18, chapters 16 and 17].

9. Conclusions

This article outlines the methodology for analytically solving the 1D governing equations of rapidly switched heat regenerators in counterflow with a uniform gradient of the matrix temperature. It has been shown that, when a Lagrange system is used, it is convenient to classify the gas particles flowing within the regenerator as 'cold', 'hot' and 'internal' particles according to Organ's concept of independent flow regimes. Also, the definition of the so-called cold and hot zones of the matrix gives insight when the gas energy equation is integrated along the path of a selected gas particle.

It was found that the phenomenon of the flush phase is able to affect positively and considerably the heat transfer performance of a rapidly switched regenerator, especially when the number of transfer units is low and the flush ratio is high. In fact, when the blow period (i.e. the heating period of the fluid) starts, the fluid gives up heat to the matrix, and only subsequently absorbs heat from it. Similarly, when the reverse period (i.e. the cooling period of the fluid) starts, the fluid absorbs heat from the matrix, and only subsequently supplies heat to it. Also, it was found that the one-dimensional, cyclically reversing flow in a regenerator is inevitably accompanied by discontinuities in the lengthwise distribution of temperature.

Finally, an easy-to-handle expression for the regenerator effectiveness as a function of NTU and flush ratio has been provided. It indicates that in rapid cyclic flow situations (typical of Stirling regenerators) the effectiveness is underestimated by the conventional regenerator theory as it fully neglects the flush phase. A comparison with numerical results of Organ's treatment has shown an excellent agreement, in particular for high values of N_{TU}. The investigation of the pressure fluctuations remains a subject for future research.

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